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# Planck Absolute Entropy of the Kerr Black Hole

**Zhao Zheng**[1]

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The thermal character of the inner horizon of the Kerr black hole is studied. There is a quantum thermal effect, "Hawking absorption," near the inner horizon. We give a new formulation of the Bekenstein–Smarr formula and redefine the entropy of the black hole. The redefined entropy must go to zero as the temperature of the black hole approaches absolute zero. The entropy satisfies the Nernst theorem, so it can be regarded as the Planck absolute entropy of the Kerr black hole.

## 1. INTRODUCTION

A remarkable relationship was established between thermodynamics and black hole physics when the area theorem and the Bekenstein–Smarr formula were discovered. The area of the event horizon (outer horizon) of a black hole is regarded as the entropy (Hawking, 1972) and the surface gravity of the horizon is regarded as the temperature of the black hole (Bardeen *et al.*, 1973). Four laws of black hole mechanics were given (Bardeen *et al.*, 1973; Bekenstein, 1972; Smarr, 1973) which were supported by the discovery of Hawking radiation (Hawking, 1975). The Hawking effect on the temperature and thermal radiation of a black hole and the "generalized second law" on black hole entropy and ordinary thermodynamic entropy are well known.

However, a problem on black hole entropy is still open (Unruh and Wald, 1982; Frolov and Page, 1993; Wald, 1994). The Nernst formulation of the third law of ordinary thermodynamics (often referred to as the Nernst theorem) asserts that the entropy of a system must go to zero as its temperature goes to zero. This assertion is commonly considered to be a fundamental law of thermodynamics. But the entropy of a black hole does not go to zero as its temperature approaches absolute zero (Lee *et al.*, 1996; Wald, 1998). The entropy $S$ of a Kerr black hole is given by

[1] Department of Physics, Beijing Normal University, Beijing 100875, China.



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$$S = K_B A/4 = \pi(r_+^2 + a^2)K_B = 2\pi M r_+ K_B \tag{1}$$

where $M$ and $a$ denote, respectively, the mass and angular momentum per unit mass of the black hole (Wald, 1994). $A$ and $r_+$ are, respectively, the area and radius of the event horizon of the hole. $K_B$ is Boltzmann's constant. Thus, absolute zero temperature corresponds to the "extremal black hole," $a = M$ and $r_+ = M$. The entropy at absolute zero temperature is thus

$$S = 2\pi M^2 K_B \tag{2}$$

which is nonvanishing and therefore has a functional dependence on the state parameter $M$ or $a$. Thus, the Kerr black holes violate the black hole analog of the "Nernst theorem." Therefore, it has been held that the entropy of black hole is not the Planck absolute entropy (Lee *et al.*, 1996; Wald, 1998).

In this paper, we present a new idea. We assert that the thermal character of Kerr black holes should be determined not only by the parameters of the outer horizon $r_+$ (event horizon), but also by the parameters of the inner horizon $r_-$ (Cauchy horizon). A Kerr black hole is a thermodynamic system composed of two subsystems, its outer horizon and its inner horizon. The entropy of a Kerr black hole depends on the area of the outer horizon and minus the area of the inner horizon. In Section 2, we study the thermal character of the inner horizon and point out that there exists a quantum effect, "Hawking absorption." In Section 3 we rewrite the Bekenstein–Smarr formula and redefine the entropy of the black hole. The redefined entropy satisfies the Nernst theorem. It can be regarded as the Planck absolute entropy of Kerr black holes. Section 4 is a conclusion and discussion.

## 2. HAWKING ABSORPTION AND THERMAL CHARACTER OF INNER HORIZON

Hawking and others proved that the outer horizon of a stationary black hole generates thermal radiation whose temperature is proportional to surface gravity $\kappa$ of the black hole. He explained the mechanism of the radiation as follows. The Killing vector $(\partial/\partial t)^a$ is spacelike in the "one-way membrane" region inside the black hole. Thus, a negative-energy particle may exist there. The particle–antiparticle pairs created by vacuum fluctuation outside the horizon and near it may be realized by the tunnel effect (Hawking, 1975). The negative-energy antiparticle falls into the black hole and travels forward in time to the singularity, and the positive-energy particle escapes from the horizon to infinity. In fact, the negative-energy antiparticle traveling forward in time toward the singularity is equivalent to a positive-energy particle traveling in the reversed time out of the singularity toward the event horizon. The particle is scattered by the horizon, then travels forward in time to infinity.

This explanation is very nice for the Hawking radiation of a Schwarzschild black hole, but it has some difficulties for Kerr black holes and Kerr–Newman black holes, because the "one-way membrane" region is only located between the outer horizon $r_+$ and the inner horizon $r_-$, $r_- < r < r_+$, where negative-energy particles and antiparticles may exist. The region inside the inner horizon ($r < r_-$) is not a "one-way membrane," where negative-energy particles and antiparticles cannot travel forward in time transiting the region ($r < r_-$) to the singularity. One has the problem of explaining the Hawking radiation of Kerr and Kerr–Newman black holes.

We explain it in terms of negative-energy antiparticles traveling in the reversed time direction from the inner horizon to the singularity. They are equivalent to positive-energy particles traveling forward in time from the singularity to the inner horizon. At the inner horizon, the positive-energy particles arriving forward in time from the singularity cancel out the negative-energy antiparticles arriving in the reversed time direction from the outer horizon and transit the "one-way membrane" region. Thus, we see that the inner horizon introduces radiation from the singularity. The radiation will be absorbed by the inner horizon. This quantum effect can be named "Hawking absorption." Thus, we can explain the Hawking radiation of Kerr black holes and Kerr–Newman black holes. There is a flow of positive-energy particles produced near the singularity, which propagate forward in time and arrive at the inner horizon. These particles are scattered by the inner horizon, then go in the reversed time toward the outer horizon, transiting the "one-way membrane" region. At the outer horizon, these particles are scattered again, then travel forward in time to infinity.

Now, let us consider the quantum effect near the inner horizon. We shall show that it is a thermal effect, and that there does exist "Hawking absorption" at the inner horizon.

First, we give the surface gravity of the outer horizon, $\kappa_+$, and the surface gravity of the inner horizon, $\kappa_-$. They are defined as (Zhao, 1981; Zhao *et al.*, 1981)

$$\begin{aligned}\kappa_\pm &= \lim_{r\to r_\pm}\left(\pm b\,\frac{dt}{d\tau}\right) = \lim_{r\to r_\pm}\left(\mp\frac{1}{2}\,\sqrt{\frac{-g^{11}}{g^{00}}}\,\frac{(g^{00})'}{g^{00}}\right)\\ &= \lim_{r\to r_\pm} \pm\frac{1}{2(r-r_\pm)}\,\sqrt{\frac{-g^{11}}{g^{00}}}\end{aligned} \tag{3}$$

where $b = \sqrt{g_{11}}\, d^2r/ds^2$ is the proper acceleration of a particle which rests outside the horizon and near it. $dt/d\tau$ is the redshift factor, and $g_{\mu\nu}$ is the Kerr metric. We have

$$\kappa_{\pm} = \lim_{r \to r_{\pm}} \pm \frac{1}{2(r - r_{\pm})} \left\{ \frac{\Delta}{\rho^2} \frac{\Delta\rho^2}{[(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta]} \right\}^{1/2} = \frac{r_+ - r_-}{2(r_{\pm}^2 + a^2)} \tag{4}$$

where

$$\Delta = r^2 + a^2 - 2Mr = (r - r_+)(r - r_-)$$
$$\rho^2 = r^2 + a^2 \cos^2\theta, \qquad r_{\pm} = M \pm (M^2 - a^2)^{1/2}$$

It should be noticed that the outer horizon of the Kerr black hole is a future horizon for the observer outside the hole ($r > r_+$), but the inner horizon is a "past horizon" for the observer inside the hole ($r < r_-$). It means that the inner horizon is a horizon of a white hole for the observer in the region $r < r_-$. The physical process near the white hole is a time reversal of the physical process near the black hole. There is Hawking radiation for black holes, so we expect "Hawking absorption" for white holes. Therefore, there is "Hawking absorption" to the inner horizon of Kerr black holes for the observers inside the hole ($r < r_-$).

In the Kerr space-time, the radical equation of the Klein–Gordon equation can be reduced to

$$\Delta \frac{d^2\phi}{dr^2} + 2(r - M)\frac{d\phi}{dr} = \left(\lambda^2 + \mu^2 r^2 - \frac{K^2}{\Delta}\right)\phi \tag{5}$$

where $K = (r^2 + a^2)\omega - ma$, and $\phi$, $\mu$, and $\omega$ are, respectively, wave function, mass, and angular momentum of Klein–Gordon particles (Zhao *et al.*, 1981). $\lambda$ is a constant from separating variables. Introducing the tortoise coordinate transformation

$$\begin{cases} \dfrac{d\hat{r}}{dr} = \pm\dfrac{r^2 + a^2}{\Delta} \\ \hat{r} = \pm\left[r + \dfrac{M}{\sqrt{M^2 - a^2}}\left(r_+ \ln\dfrac{|r - r_+|}{r_+} - r_- \ln\dfrac{|r - r_-|}{r_-}\right)\right] \end{cases} \tag{6}$$

where "+" is for $r > r_+$ and "−" is for $r < r_-$.

Thus, near the horizons, the Klein–Gordon equation (5) can be reduced to

$$\frac{d^2\phi}{d\hat{r}^2} + (\omega - \omega_0)^2\,\phi = 0 \tag{7}$$

where $\omega_0 = m\Omega_{\pm}$, and $\Omega_+$ and $\Omega_-$ are, respectively, the angular velocity of the outer horizon and the inner horizon,

$$\Omega_\pm = a/(r_\pm^2 + a^2) \tag{8}$$

Studying Eq. (7) near the outer horizon, we can prove that there is Hawking radiation emitted from the horizon. But now we are only interested in the case near the inner horizon. We can get the solution of Eq. (7) as

$$\phi = e^{\pm i(\omega - \omega_0)\hat{r}} \tag{9}$$

Thus, we have the outgoing wave

$$\phi_{\text{out}} = \exp[-i\omega t + i(\omega - \omega_0)\hat{r}] = \exp\left[-i\omega\left(t - \frac{\omega - \omega_0}{\omega}\hat{r}\right)\right] = e^{-i\omega u} \tag{10}$$

and the ingoing wave

$$\begin{aligned}\phi_{\text{in}} &= \exp[-i\omega t - i(\omega - \omega_0)\hat{r}] = \exp\left[-i\omega\left(t + \frac{\omega - \omega_0}{\omega}\hat{r}\right)\right] \\ &= e^{-i\omega u}e^{-2i(\omega-\omega_0)\hat{r}}\end{aligned} \tag{11}$$

Because the inner horizon can be regarded as a “past horizon” by an observer inside the black hole where $r < r_-$, we adopt the retarded Eddington–Finkelstein coordinate $u = t - [(\omega - \omega_0)/\omega]\hat{r}$.

When $r \to r_-$, we have $\hat{r} \to -\infty$. When $r \to 0$, we get $\hat{r} \to 0$. Therefore, Eq. (10) is just the outgoing wave emitted by the inner horizon. Nevertheless, Eq. (11) represents the ingoing wave to the inner horizon. It is easy to see that

$$\hat{r} \sim \frac{1}{2\kappa_-}\ln(r_- - r) \tag{12}$$

as $r \to r_-$. Thus, we have

$$\phi_{\text{in}} = e^{-i\omega u}\,(r_- - r)^{-i(\omega-\omega_0)/\kappa_-} \tag{13}$$

Because $\phi_{\text{in}}$ is not analytic at the inner horizon, we can analytically extend it, around the inner horizon $r_-$ along the upper semicircle of radius $|r - r_-|$ in the complex $r$ plane, into the “one-way membrane” region ($r_- < r < r_+$), as $|r_- - r|e^{i\pi} = (r - r_-)e^{i\pi}$. Thus, we get $\phi_{\text{in}}$ in the region $r_- < r < r_+$,

$$\begin{aligned}\phi_{\text{in}} &= e^{-i\omega u}[(r - r_-)e^{i\pi}]^{-i(\omega-\omega_0)/\kappa_-} = e^{-i\omega u}(r - r_-)^{-i(\omega-\omega_0)/\kappa_-}e^{\pi(\omega-\omega_0)/\kappa_-} \\ &= \phi_i'(r - r_-)e^{\pi(\omega-\omega_0)/\kappa_-}\end{aligned} \tag{14}$$

The total wave function is

$$\phi_\omega = N_\omega[Y(r_- - r)\phi_{\text{in}}(r_- - r) + e^{\pi(\omega-\omega_0)/\kappa_-}Y(r - r_-)\phi'_{in}(r - r_-)] \tag{15}$$

where

$$Y(r) = \begin{cases} 1, & r \geq 0 \\ 0, & r < 0 \end{cases} \tag{16}$$

$$\phi'_{\text{in}} = e^{-i\omega u}(r - r_-)^{-i(\omega-\omega_0)/\kappa_-} = e^{-i\omega u}e^{-2i(\omega-\omega_0)\hat{r}}$$

We have

$$(\phi_\omega, \phi_\omega) = N_\omega^2(1 \pm e^{(\omega-\omega_0)/K_B T_-}) = \pm 1 \tag{17}$$

Thus, we get the result that the inner horizon absorbs thermal radiation from the region $r < r_-$, whose thermal spectrum and temperature are

$$N_\omega^2 = \frac{1}{(\omega-\omega_0)/e^{K_B T_-} \pm 1} \tag{18}$$

$$T_- = \frac{\kappa_-}{2\pi K_B} \tag{19}$$

Thus, we have proved that there exists some radiation from the region $r < r_-$ to the inner horizon, which is thermal radiation whose temperature is $T_-$. We can name the effect of the inner horizon absorbing blackbody radiation "Hawking absorption." Because the Kerr black hole is stationary, its outer horizon is in thermal equilibrium with the thermal radiation outside the black hole. Its inner horizon is certainly in thermal equilibrium with the thermal radiation in the region $r < r_-$. Therefore, the inner horizon not only absorbs thermal radiation at temperature $T_-$, but also emits thermal radiation at the same temperature $T_-$. So, the inner horizon of the Kerr black hole can be regarded as a thermodynamic system whose temperature is $T_-$.

Thus, we have overcome the difficulty explaining the Hawking radiation of a Kerr black hole. The Hawking radiation of the black hole originates from thermal radiation near the singularity. The thermal radiation is absorbed by the inner horizon, then travels in the reversed time direction, transiting the "one-way membrane" region and arriving at the outer horizon. It is scattered by the outer horizon, then travels forward in time to infinity as Hawking radiation.

We can calculate the area of the outer horizon $A_+$ and the area of the inner horizon $A_-$ as

$$A_\pm = \pm \int \sqrt{g}\, d\theta\, d\varphi = \pm 4\pi(r_\pm^2 + a^2) \tag{20}$$

where $g = (r_\pm^2 + a^2)^2 \sin^2\theta$ is the determinant of the 2-dimensional metric on the inner and outer horizons. Because the inner horizon is like the horizon of a white hole, we define $A_-$ as minus. We will see that $A_-$ gives a contribution, as $A_+$ does, to the entropy of the black hole.

## 3. NEW FORMULATION OF THE BEKENSTEIN–SMARR FORMULA AND THE PLANCK ABSOLUTE ENTROPY OF A BLACK HOLE

Using Eqs. (4), (8), and (20), it is easy to find

$$2\Omega_+ J = r_- = M - \sqrt{M^2 - a^2} \tag{21}$$

$$\frac{1}{4\pi}\kappa_+ A_+ = \sqrt{M^2 - a^2} \tag{22}$$

Adding Eq. (21) to (22), we get the Bekenstein–Smarr integral formula

$$M = \frac{1}{4\pi}\kappa_+ A_+ + 2\Omega_+ J \tag{23}$$

which contains only the parameters of the outer horizon. Similarly, we obtain

$$2\Omega_- J = r_+ = M + \sqrt{M^2 - a^2} \tag{24}$$

$$\frac{1}{4\pi}\kappa_- A_- = -\sqrt{M^2 - a^2} \tag{25}$$

Thus, another formulation of the Bekenstein–Smarr integral formula which contains only the parameters of inner horizon can be given as

$$M = \frac{1}{4\pi}\kappa_- A_- + 2\Omega_- J \tag{26}$$

Combining Eq. (23) with (26), we get

$$M = \frac{1}{8\pi}\kappa_+ A_+ + \Omega_+ J + \frac{1}{8\pi}\kappa_- A_- + \Omega_- J \tag{27}$$

This is also a new formulation of the Bekenstein–Smarr integral formula, which contains parameters of both the inner and outer horizons. Equations (23), (26), and (27) are equivalent to each other.

Differentiating with respect to Eq. (23), we have

$$\delta M = \frac{1}{4\pi}\kappa_+ \delta A_+ + \frac{1}{4\pi} A_+ \delta\kappa_+ + 2\Omega_+ \delta J + 2J\delta\Omega_+ \tag{28}$$

It is easy to verify

$$2J\delta\Omega_+ = \frac{Ma}{r_+\sqrt{M^2 - a^2}}\,\delta a - \frac{a^2}{M\sqrt{M^2 - a^2}}\,\delta M \tag{29}$$

$$\frac{1}{4\pi} A_+\delta\kappa_+ = \left(\frac{a^2}{M\sqrt{M^2 - a^2}} - 1\right)\delta M - \frac{Ma}{r_+\sqrt{M^2 - a^2}}\,\delta a \tag{30}$$

Substituting Eqs. (29) and (30) into (28), we get the Bekenstein–Smarr differential formula

$$\delta M = \frac{1}{8\pi}\kappa_+\delta A_+ + \Omega_+\delta J \tag{31}$$

Similarly, differentiating Eq. (26), we have

$$\delta M = \frac{1}{4\pi}\kappa_-\delta A_- + \frac{1}{4\pi} A_-\delta\kappa_- + 2\Omega_-\delta J + 2J\delta\Omega_- \tag{32}$$

It can be verified that

$$2J\delta\Omega_- = \frac{-Ma}{r_-\sqrt{M^2 - a^2}}\,\delta a + \frac{a^2}{M\sqrt{M^2 - a^2}}\,\delta M \tag{33}$$

$$\frac{1}{4\pi} A_-\delta\kappa_- = \left(-1 - \frac{a^2}{M\sqrt{M^2 - a^2}}\right)\delta M + \frac{Ma}{r_-\sqrt{M^2 - a^2}}\,\delta a \tag{34}$$

Substituting Eqs. (33) and (34) into Eq. (32), we get another Bekenstein–Smarr differential formula

$$\delta M = \frac{1}{8\pi}\kappa_-\delta A_- + \Omega_-\delta J \tag{35}$$

which contains only the parameters of the inner horizon. It is easy to verify that there is a new BS differential formula corresponding to Eq. (27), as follows:

$$\delta M = \frac{1}{16\pi}\kappa_+\delta A_+ + \Omega_+\delta J_+ + \frac{1}{16\pi}\kappa_-\delta A_- + \Omega_-\delta J_- \tag{36}$$

where

$$J_+ = J_- = J/2 \tag{37}$$

Equations (31), (35), and (36) are mathematically equivalent. Equation (36) can be rewritten as

$$\delta M = T_+\delta S_+ + \Omega_+\delta J_+ + T_-\delta S_- + \Omega_-\delta J_- \tag{38}$$

where

$$T_{\pm} = \kappa_{\pm}/2\pi K_B, \qquad S_{\pm} = K_B A_{\pm}/8 \tag{39}$$

They are, respectively, the temperature and the entropy of the inner and outer horizons.

Now, we have equivalently rewritten the usual BS differential formula containing only the parameters of the outer horizon, Eq. (31), to the new formulation (38). Equation (38) is different from (31) in physical content. In Eq. (31), the Kerr black hole is regarded as a single thermodynamic system only composed of the outer horizon, but in Eq. (38), the black hole is regarded as a complex thermodynamic system composed of two subsystems, the outer horizon and the inner horizon. The temperature of the outer horizon is just the usual black hole temperature given by $\kappa_+$ in Eq. (4). The temperature of the inner horizon is the same as that given by $\kappa_-$ in Eq. (19).

It should be noticed that Eqs. (38) and (39) mean that both the outer horizon and inner horizon contribute to the entropy of the black hole,

$$\begin{aligned}\tilde{S} &= S_+ + S_- = (K_B/8)(A_+ + A_-)\\ &= (\pi/2)(r_+^2 - r_-^2)K_B = 2\pi M\sqrt{M^2 - a^2}K_B\end{aligned} \tag{40}$$

This is different from Eq. (1), which contains only the area of outer horizon, as

$$S = K_B A_+/4 = 2\pi M r_+ K_B = 2\pi M(M + \sqrt{M^2 - a^2})K_B \tag{41}$$

It is easy to see that the temperatures of the inner and outer horizons go to absolute zero when the Kerr black hole approaches the extremal one, $M \to a$, $r_+ = r_- = M$,

$$T_{\pm} = \kappa_{\pm}/2\pi K_B = \frac{1}{2\pi K_B}\frac{r_+ - r_-}{2(r_{\pm}^2 + a^2)} \to 0 \tag{42}$$

The black hole entropy $S$ given by Eq. (41) does not vanish, but $\tilde{S}$ given by Eq. (42) approaches zero. We see that the black hole entropy $\tilde{S}$ redefined by us satisfies the Nernst theorem. It can be regarded as the Planck absolute entropy of the black hole.

## 4. CONCLUSION AND DISCUSSION

We have proved that the inner horizon of the Kerr black hole has a thermal character and a quantum effect called "Hawking absorption." Furthermore, we have explained the Hawking radiation mechanism of the Kerr black hole.

The black hole is regarded as a complex thermodynamic system composed of two subsystems, the inner horizon and outer horizon. Both the Bekenstein–Smarr integral formula and the differential formula are rewritten. The new formulas are mathematically equivalent with the old formulas, but are different in physical content. The temperature of the outer horizon given by the new formulas is the same as the usual one. The temperature of the inner horizon is the same as that of "Hawking absorption."

The black hole entropy given by the new formulas is different from the well-known one that contains the area of the outer horizon only. The new entropy contains contributions from the areas of both the inner and outer horizons. The new entropy vanishes when the temperature of the black hole goes to absolute zero. Thus the new entropy satisfies the Nernst theorem. It can be regarded as the Planck absolute entropy of the Kerr black hole.

The formula for the black hole entropy given only by the area of the outer horizon is not complete: The correct formula given in this paper contains contributions from both the outer and inner horizons.

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